Financial Economics
3: Uncertainty

Stefano Lovo

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Today the price of one share of Total is $P_0 = Eu\ 39$. You buy 1000 share of Total. After one year

- you receive the dividends \( \tilde{D} \) payed by Total.
- you resell the shares of Total at price \( \tilde{P}_1 \).

What will the annual rate of return of your investment be?

\[
\tilde{r} := \frac{\tilde{P}_1 + \tilde{D} - P_0}{P_0}
\]

- Capital gain/loss := \( \frac{\tilde{P}_1 - P_0}{P_0} \)
- Yield component := \( \frac{\tilde{D}}{P_0} \)
Example

Today the price of one share of Total is $P_0 = Eu\ 39$. You buy 1000 share of Total. After one year

- you receive the dividends $Eu\ 4$ per-share.
- you resell the shares of Total at price $Eu\ 38$.

What is the annual rate of return of your investment?
Example

Today the price of one share of Total is $P_0 = Eu 39$. You buy 1000 share of Total. After one year

- you receive the dividends $Eu 4$ per-share.
- you resell the shares of Total at price $Eu 38$.

What is the annual rate of return of your investment?

$$r = \frac{38 + 4 - 39}{39} = 7.7\% = -2.6\% + 10.3\%$$

- Capital gain/loss : $\frac{38 - 39}{39} = -2.6\%$
- Yield component : $\frac{4}{39} = 10.3\%$
When an investor buys a stock:
- He knows the price he pays.
- He does not know the dividend he will receive.
- He does not know the resale price.

Example

The actual rate of return of a stock depends on performance of the company and of the whole economy.

You spent Eu100 to buy 1 share of XYZ Corp.

<table>
<thead>
<tr>
<th>Event</th>
<th>$P_1$</th>
<th>$D$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recession</td>
<td>Eu 80</td>
<td>Eu 0</td>
<td>−20%</td>
</tr>
<tr>
<td>Normal</td>
<td>Eu 103</td>
<td>Eu 7</td>
<td>10%</td>
</tr>
<tr>
<td>Boom</td>
<td>Eu 110</td>
<td>Eu 10</td>
<td>20%</td>
</tr>
</tbody>
</table>
Contingent return rates

Example

<table>
<thead>
<tr>
<th>Event</th>
<th>Probability</th>
<th>Alcatel</th>
<th>BNP</th>
<th>BMW</th>
<th>...</th>
<th>Treasury bill</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recession</td>
<td>0.7</td>
<td>−10%</td>
<td>−5%</td>
<td>−15%</td>
<td>...</td>
<td>1.5%</td>
</tr>
<tr>
<td>Normal</td>
<td>0.25</td>
<td>12%</td>
<td>10%</td>
<td>20%</td>
<td>...</td>
<td>1.5%</td>
</tr>
<tr>
<td>Boom</td>
<td>0.05</td>
<td>18%</td>
<td>30%</td>
<td>22%</td>
<td>...</td>
<td>1.5%</td>
</tr>
</tbody>
</table>
Let $\Omega$ be the set of all possible events or scenarios. Let $\omega \in \Omega$ be one event.

**Definition**

The **probability** of an event $\omega$, is the likelihood that the event will happen, it is denoted $Pr(\omega)$ and satisfies:

1. $0 \leq Pr(\omega) \leq 1.$
2. If $\omega$ is certain, then $Pr(\omega) = 1.$
3. If $\omega$ is impossible, then $Pr(\omega) = 0.$
4. If two events $\omega$ and $\omega'$ are mutually exclusive, then 
   \[ Pr(\omega \text{ or } \omega') = Pr(\omega) + Pr(\omega') \]
5. $Pr(\Omega) = 1$
What is the probability that the economy is not in recession?

What is the probability that the rate of return of Alcatel is less than 5%?

What is the probability that the rate of return of BMW is positive and the rate of return of BNP is negative?
Random variable

Definition
A *random variable* is a function that maps the set of possible events into a real number: \( \tilde{r} : \Omega \rightarrow \mathbb{R} \)

Example
\( r_{BMW}(\omega_1) = -15\%; \ r_{BMW}(\omega_2) = 20\%; \ r_{BMW}(\omega_3) = 22\% \)

Definition
The *distribution function* of a random variable represents the probability that the realization of a random variable reaches a given value. \( \pi_{\tilde{r}} : \mathbb{R} \rightarrow [0, 1] \)

Example
\( \pi_{BMW}(-15\%) = 0.7; \ \pi_{BMW}(20\%) = 0.25; \ \pi_{BMW}(22\%) = 0.05 \)
**Assumption:** There is a finite number $n$ of possible events (or state of the economy)

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$$

**Notation:**

$$\pi_i := Pr(\omega_i)$$

$$r_{A,i} := \tilde{r}_A(\omega_i)$$

with

$$\tilde{r}_A \in \{r_{A,1}, r_{A,2}, \ldots, r_{A,i}, \ldots, r_{A,n}\}$$
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- $r_{Alcatel,2} = ?$
- $\pi_1 = ?$
- $Pr(\tilde{r}_{Alcatel} > 0) = ?$
**Expectation of a random variable**

How can we make prediction about the rate of return of an asset?

**Definition**

The **expected return** of an asset with return rate $\tilde{r}$ is:

$$E[\tilde{r}] = \sum_{i=1}^{n} \pi_i r_i = \pi_1 r_1 + \pi_2 r_2 + \ldots + \pi_n r_n$$

**Example**

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$$E[\tilde{r}_{\text{Alcatel}}] = 0.7(-10\%) + 0.25 \times 12\% + 0.05 \times 18\% = -3.1\%$$
The expectation is a linear operator

Let $\tilde{r}_A$ and $\tilde{r}_B$ be two random variables and let $\alpha$ and $\beta$ be two real numbers. Then

$$E \left[ \alpha \tilde{r}_A + \beta \tilde{r}_B \right] = \alpha E[\tilde{r}_A] + \beta E[\tilde{r}_B]$$

Proof:

$$E \left[ \alpha \tilde{r}_A + \beta \tilde{r}_B \right] = \sum_{i=1}^{n} \pi_i (\alpha r_{A,i} + \beta r_{B,i}) =$$

$$= \sum_{i=1}^{n} \pi_i (\alpha r_{A,i}) + \sum_{i=1}^{n} \pi_i (\beta r_{B,i}) = \alpha \sum_{i=1}^{n} \pi_i r_{A,i} + \beta \sum_{i=1}^{n} \pi_i r_{B,i} =$$

$$= \alpha E[\tilde{r}_A] + \beta E[\tilde{r}_B]$$

Example

Let $E[\tilde{r}_A] = 12\%$ and $E[\tilde{r}_B] = 33\%$. Then

$$E \left[ \frac{1}{3} \tilde{r}_A + \frac{2}{3} \tilde{r}_B \right] = \frac{1}{3} 12\% + \frac{2}{3} 33\% = 26\%$$
Is the expected return a good predictor of the actual return?

Example

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<td>-8%</td>
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$$E[\tilde{r}_A] = E[\tilde{r}_B] = E[\tilde{r}_C] = 5\%$$

Expectation is a perfect forecast for $\tilde{r}_A$, a good forecast for $\tilde{r}_B$ and a vague forecast for $\tilde{r}_C$.

**Question:** How can we measure the quality of the expected return as forecast for the realized return?
If I predict that the actual return on C will be $E[\tilde{r}_C]$, how far should I expect to be my prediction from the true $r_C$?

Definition
The variance of a random variable $\tilde{r}$ is equal to
\[
\text{Var}(\tilde{r}) := E[(\tilde{r} - E[\tilde{r}])^2]
\]
\[
= n \sum_{i=1}^{n} \pi_i (r_i - E[\tilde{r}])^2
\]

Definition
The standard deviation of a random variable $\tilde{r}$ is equal to
\[
\sigma_{\tilde{r}} = \sqrt{\text{Var}(\tilde{r})}
\]

Interpretation: Variance and standard deviation measure the dispersion of realized return around the expected return.
If I predict that the actual return on C will be $E \left[ \tilde{r}_C \right]$, how far should I expect to be my prediction from the true $r_C$?

**Definition**

The **variance** of a random variable $\tilde{r}$ is equal to

$$Var \left[ \tilde{r} \right] : = E \left[ (\tilde{r} - E [\tilde{r}])^2 \right] = \sum_{i=1}^{n} \pi_i (r_i - E [\tilde{r}])^2$$

**Interpretation:** Variance and standard deviation measure the dispersion of realized return around the expected return.
If I predict that the actual return on $C$ will be $E[\tilde{r}_C]$, how far should I expect to be my prediction from the true $r_C$?

Definition

The variance of a random variable $\tilde{r}$ is equal to

$$\text{Var} [\tilde{r}] := E \left[ (\tilde{r} - E[\tilde{r}])^2 \right] = \sum_{i=1}^{n} \pi_i \left( r_i - E[\tilde{r}] \right)^2$$

Definition

The standard deviation of a random variable $\tilde{r}$ is equal to

$$\sigma_{\tilde{r}} = \sqrt{\text{Var} [\tilde{r}]}$$

Interpretation: Variance and standard deviation measure the dispersion of realized return around the expected return.
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## Variance and Standard Deviation: example

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$$Var[\tilde{r}] = 0.5(r_1 - E[\tilde{r}])^2 + 0.5(r_2 - E[\tilde{r}])^2$$
Variance and Standard Deviation: example

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\[
\text{Var} \left[ \tilde{r} \right] = 0.5 (r_1 - E [\tilde{r}])^2 + 0.5 (r_2 - E [\tilde{r}])^2
\]

\[
\text{Var} \left[ \tilde{r}_A \right] = 0.5 (0.05 - 0.05)^2 + 0.5 (0.05 - 0.05)^2 = 0
\]

\[
\sigma_{\tilde{r}_A} = \sqrt{0} = 0
\]
Variance and Standard Deviation: example

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$$Var[\tilde{r}] = 0.5(r_1 - E[\tilde{r}])^2 + 0.5(r_2 - E[\tilde{r}])^2$$

$$Var[\tilde{r}_A] = 0.5(0.05 - 0.05)^2 + 0.5(0.05 - 0.05)^2 = 0$$

$$\sigma_{\tilde{r}_A} = \sqrt{0} = 0$$

$$Var[\tilde{r}_B] = 0.5(0.055 - 0.05)^2 + 0.5(0.045 - 0.05)^2 = 0.000025$$

$$\sigma_{\tilde{r}_B} = \sqrt{0.000025} = 0.005 = 0.5\%$$
**Example**

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$$\text{Var} [\tilde{r}] = 0.5(r_1 - E [\tilde{r}])^2 + 0.5(r_2 - E [\tilde{r}])^2$$

$$\text{Var} [\tilde{r}_A] = 0.5(0.05 - 0.05)^2 + 0.5(0.05 - 0.05)^2 = 0$$

$$\sigma_{\tilde{r}_A} = \sqrt{0} = 0$$

$$\text{Var} [\tilde{r}_B] = 0.5(0.055 - 0.05)^2 + 0.5(0.045 - 0.05)^2 = 0.000025$$

$$\sigma_{\tilde{r}_B} = \sqrt{0.000025} = 0.005 = 0.5\%$$

$$\text{Var} [\tilde{r}_C] = 0.5(0.18 - 0.05)^2 + 0.5(-0.08 - 0.05)^2 = 0.0169$$

$$\sigma_{\tilde{r}_C} = \sqrt{0.0169} = 13\%$$
Properties of the variance

1. \[ \text{Var} \left[ r \right] = E \left[ r^2 \right] - E \left[ r \right]^2 \]

2. If \( k \) is a constant, then
\[ \text{Var} \left[ k \tilde{r} \right] = k^2 \text{Var} \left[ r \right] \]

3. Let \( \tilde{r}_A \) and \( \tilde{r}_B \) be two random variables and \( \alpha \) and \( \beta \) two real numbers, then
\[ \text{Var} \left[ \alpha \tilde{r}_A + \beta \tilde{r}_B \right] = \alpha^2 \text{Var} \left[ \tilde{r}_A \right] + \beta^2 \text{Var} \left[ \tilde{r}_B \right] + 2\alpha\beta \text{Cov} \left[ \tilde{r}_A, \tilde{r}_B \right] \]

where \( \text{Cov} \left[ \tilde{r}_A, \tilde{r}_B \right] \) is the covariance between \( \tilde{r}_A \) and \( \tilde{r}_B \).
Covariance

Definition

The covariance between two random variables \( \tilde{r}_A \) and \( \tilde{r}_B \) is equal to

\[
\text{Cov} [\tilde{r}_A, \tilde{r}_B] : = E \left[ (\tilde{r}_A - E[\tilde{r}_A]) (\tilde{r}_B - E[\tilde{r}_B]) \right] = \\
= \sum_{i=1}^{n} \pi_i (r_{A,i} - E[\tilde{r}_A]) (r_{B,i} - E[\tilde{r}_B])
\]

Interpretation: The covariance measures the degree to which two random variables move together.

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Uncertainty
Positive covariance: example

If $\text{Cov} \left[ \tilde{r}_A, \tilde{r}_B \right] > 0$, then it is likely that when $\tilde{r}_A > E \left[ \tilde{r}_A \right]$, also $\tilde{r}_B > E \left[ \tilde{r}_B \right]$

Example

Evolution of the stock prices of Renault and Michelin
Negative covariance: example

- If $\text{Cov} \left[ \tilde{r}_A, \tilde{r}_B \right] < 0$, then it is likely that while $\tilde{r}_A > E [\tilde{r}_A]$, $\tilde{r}_B < E [\tilde{r}_B]$

Example

Rate of return of Gold and CAC40
Properties of the Covariance

1. \( \text{Cov} [\tilde{r}_A, \tilde{r}_B] = E [\tilde{r}_A \tilde{r}_B] - E [\tilde{r}_A] E [\tilde{r}_B] \)

2. \( \text{Cov} [\tilde{r}_A, \tilde{r}_A] = \text{Var} [\tilde{r}_A] \)

3. \( \text{Cov} [\tilde{r}_A, \tilde{r}_B] = \text{Cov} [\tilde{r}_B, \tilde{r}_A] \)

4. If \( \alpha \) and \( \beta \) are two real numbers, then \( \text{Cov} [\tilde{r}_C, \alpha \tilde{r}_A + \beta \tilde{r}_B] = \alpha \text{Cov} [\tilde{r}_C, \tilde{r}_A] + \beta \text{Cov} [\tilde{r}_C, \tilde{r}_B] \)
**Correlation coefficient**

**Definition**

The **correlation coefficient** between two random variables \( \tilde{r}_A \) and \( \tilde{r}_B \) is equal to

\[
\rho_{\tilde{r}_A, \tilde{r}_B} = \frac{\text{Cov} [\tilde{r}_A, \tilde{r}_B]}{\sigma_{\tilde{r}_A} \sigma_{\tilde{r}_B}}
\]

**properties:**

1. \(-1 \leq \rho_{\tilde{r}_A, \tilde{r}_B} \leq 1\)
2. If \( \rho_{\tilde{r}_A, \tilde{r}_B} = -1 \), then there exists \( \alpha < 0 \) and \( \beta \) such that
   \[
   \tilde{r}_A = \alpha \tilde{r}_B + \beta
   \]
3. If \( \rho_{\tilde{r}_A, \tilde{r}_B} = 1 \), then there exists \( \alpha > 0 \) and \( \beta \) such that
   \[
   \tilde{r}_A = \alpha \tilde{r}_B + \beta
   \]
Correlation coefficient: example

\[ \rho = 1 \]
Correlation coefficient: example

\[ \rho = -1 \]
Correlation coefficient: example

\[ \rho = 0 \]
Estimation of expected return, variance and covariance

If we do not know the distribution probabilities of return rates how can we estimate an asset expected return, variance and its covariance with another asset?
Estimation of expected return, variance and covariance

If we do not know the distribution probabilities of return rates how can we estimate an asset expected return, variance and its covariance with another asset?

- We are in year \( T + 1 \).
- Let \( R_{i,t} \) be the realized annual return of asset \( i \) in a past year \( t \).
- Suppose you observe the realized return of assets \( A \) and \( B \) during the last \( T \) years:

\[
\{ R_{A,1}, R_{A,2}, \ldots R_{A,T} \} \text{ and } \{ R_{B,1}, R_{B,2}, \ldots R_{B,T} \}
\]
Sample mean of historical returns (an estimation of $E[\tilde{r}_A]$):

$$\hat{r}_A := \frac{1}{T} \sum_{t=1}^{T} R_{A,t}$$

Sample variance of historical return (an estimation of $\text{Var}[\tilde{r}_A]$):

$$\hat{\sigma}_A^2 := \frac{1}{T - 1} \sum_{t=1}^{T} (R_{A,t} - \hat{r}_A)^2$$

Sample covariance of historical return (an estimation of $\text{Cov}[\tilde{r}_A, \tilde{r}_B]$):

$$\hat{\sigma}_{A,B} := \frac{1}{T - 1} \sum_{t=1}^{T} (R_{A,t} - \hat{r}_A) (R_{B,t} - \hat{r}_B)$$
Estimating $E[\tilde{r}_A]$, $\text{Var}[\tilde{r}_A]$ and $\text{Cov}[\tilde{r}_A, \tilde{r}_B]$

- **Sample mean of historical returns** (an estimation of $E[\tilde{r}_A]$):
  \[ \hat{r}_A := \frac{1}{T} \sum_{t=1}^{T} R_{A,t} \]

- **Sample variance of historical return** (an estimation of $\text{Var}[\tilde{r}_A]$):
  \[ \hat{\sigma}^2_A := \frac{1}{T-1} \sum_{t=1}^{T} (R_{A,t} - \hat{r}_A)^2 \]

- **Sample covariance of historical return** (an estimation of $\text{Cov}[\tilde{r}_A, \tilde{r}_B]$):
  \[ \hat{\sigma}_{A,B} := \frac{1}{T-1} \sum_{t=1}^{T} (R_{A,t} - \hat{r}_A) (R_{B,t} - \hat{r}_B) \]

**Caveat:** These estimations are not correct if the true probabilities underlying the past return rates are not stationary.