Sparse and stable Markowitz portfolios

Joshua Brodie\(^1\)    Ingrid Daubechies\(^1\)    Christine De Mol\(^2\)
Domenico Giannone\(^3\)    Ignace Loris\(^2\)

\(^1\)Princeton University, PACM
\(^2\)Université Libre de Bruxelles, ECARES and Math Dept
Vrije Universiteit Brussel, Math Dept and CAMP
\(^3\)European Central Bank (DG-R) and ECARES

Workshop "Stats in the Château", HEC, Jouy-en-Josas,
Paris, September 2009
Portfolio optimization

- $N$ securities with (stationary) returns $r_{i,t}$ at time $t$.
- $r_t = N \times 1$ vector of returns at time $t$.
- Vector of expected returns: $\mu = E[r_t]$
  Covariance matrix of returns:
  \[
  E[(r_t - \mu)(r_t - \mu)^\top] = C
  \]
- A portfolio is defined by a $N \times 1$ vector of weights $w_i$ summing to one (unit of capital): $w^\top 1_N = 1$, where $1_N$ denotes the $N \times 1$ vector of ones.
- Expected return of the portfolio: $w^\top \mu$
  Variance of the portfolio: $w^\top C w$
Markowitz portfolios

• Find a portfolio $\tilde{w}$ which has minimal variance for a given expected return $\rho = w^\top \mu$, i.e.

$$
\tilde{w} = \arg\min_w w^\top C w \\
\text{s. t. } w^\top \mu = \rho \\
w^\top 1_N = 1
$$

• Since $C = E[r_tr_t^\top] - \mu\mu^\top$, this is equivalent to

$$
\tilde{w} = \arg\min_w E \left[ |\rho - w^\top r_t|^2 \right] \\
\text{s. t. } w^\top \mu = \rho \\
w^\top 1_N = 1
$$
Our proposal

• For empirical implementation, replace expectations with sample averages and solve the following regression problem

\[
\hat{w} = \arg \min_w \frac{1}{T} \left\| \rho \mathbf{1}_T - Rw \right\|^2_2
\]

s. t. \quad w^\top \hat{\mu} = \rho
\]
\[
w^\top \mathbf{1}_N = 1,
\]

where \( \hat{\mu} = \frac{1}{T} \sum_{t=1}^T \mathbf{r}_t \) and \( R \) is the \( T \times N \) matrix of the available returns.

• Add a \( L_1 \)-norm penalty to ensure sparsity and stability.
Our sparse and stable portfolios

- Find the weights

\[ \mathbf{w}^{[\tau]} = \arg \min_{\mathbf{w}} \left[ ||\rho \mathbf{1}_T - \mathbf{Rw}||_2^2 + \tau ||\mathbf{w}||_1 \right] \] (1)

s. t. \[ \mathbf{w}^{\top} \hat{\boldsymbol{\mu}} = \rho \] (2)

\[ \mathbf{w}^{\top} \mathbf{1}_N = 1. \] (3)

where \( ||\mathbf{w}||_1 = \sum_i |w_i| \) and \( \tau \) is a positive parameter tuning the balance between the two terms.

- “Lasso” regression (Tibshirani 1996), but with extra constraints.
Why stability?

- We suspect ill-conditioning due to high collinearity is a major problem for practical implementations of the Markowitz framework and may also account for the following fact.

- It was recently shown that many portfolio constructions proposed in the literature fail to outperform the naive equally-weighted (“1/N”) portfolio. (DeMiguel, Garlappi and Uppal 2007)

- The $L_1$-norm penalty is known to regularize (stabilize) the problem. (Daubechies, Defrise and De Mol 2004)
Sparsity

- The $L_1$-norm penalty enforces sparsity of the portfolio, i.e. favors the presence of many zero weights ($\leftrightarrow$ few active assets).

- This allows for variable (asset) selection.

- Why?
Lasso regression and sparsity
Lasso regression and sparsity
Lasso regression and sparsity
Lasso regression and sparsity
Why sparsity?

- Difficulty of managing several hundreds of assets.
- The traditional Markowitz framework does not take into account transaction and monitoring costs.
- The $L_1$-norm penalty allows to cope with the transaction costs, modelling linear transaction costs and/or serving as a proxy to the $L_0$-norm (↔ sparsity, to keep control on the fixed costs)
Constrained LARS Algorithm

- The recursive LARS/homotopy algorithm allows to compute efficiently the Lasso regression solutions for all values of $\tau$, starting from the largest ones. (Osborne et al. 2000, Efron et al. 2004)

- We devised a modification of LARS able to enforce the linear constraints.

- Varying $\tau$ allows to tune the number of active positions.
Special case: No-short portfolios

- Notice $\sum_i w_i = 1 \implies \sum_i |w_i| \geq 1$.

- Limit case for $\tau$ large: $\sum_i |w_i| = 1 \iff w_i \geq 0$ for all $i$ (no short positions).

- Positive portfolios known for their good performances (Jagannathan and Ma 2003).

- But the fact that no-short portfolios are sparse seems to have gone unnoticed!
Empirical application

- We used as assets the Fama and French 48 industry portfolios (FF48) and 100 portfolios formed on size and book-to-market (FF100).

- We constructed our portfolios in June of each year from 1976 to 2006 using 5 years of historical (monthly) returns and a target return equal to the historical return of the equally-weighted portfolio.

- Performance is evaluated by out-of-sample monthly mean return $m$, standard deviation $\sigma$ and Sharpe ratio $S = \frac{m}{\sigma}$.

- Benchmark (tough!) is the evenly-weighted portfolio.
Empirical results

Performance of sparse portfolio with no short-selling (FF48)

<table>
<thead>
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<th>Evaluation period</th>
<th>( w_i \geq 0 ) for all ( i )</th>
<th>Equal weight</th>
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<tr>
<td></td>
<td>( m )</td>
<td>( \sigma )</td>
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<td>06/76-06/06</td>
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<td>41</td>
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Empirical results

Performance of sparse portfolio with no short-selling (FF100)

<table>
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<td></td>
<td>$m$</td>
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</tbody>
</table>
Empirical results FF100

Sharpe Ratio (in %)

number of active positions

Sharpe Ratio (in %)

wpos

1 5 10 15 20 25 30 35 40 45 50 55 60

11−20
21−30
31−40
41−50
51−60

w

w

w

w

w

K

bin
Generalizations

- Weighted $L_1$-norm penalty: $\sum_i s_i |w_i|$. 
- Index tracking (replace target return $\rho$ by index $y_t$). 
- Portfolio Adjustment (Rebalancing).
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Working paper (July 2007), ECORE DP2007/61, CEPR DP6474,
Published in PNAS 2009 106:12267-12272
(No 30, 28 July 2009)